

Bodily tides near the 1:1 spin-orbit resonance: correction to Goldreich's dynamical model

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Received: 9 April 2012 / Revised: 21 August 2012 / Accepted: 6 October 2012 /
Published online: 22 November 2012
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Abstract Spin-orbit coupling is often described in an approach known as “*the MacDonald torque*”, which has long become the textbook standard due to its apparent simplicity. Within this method, a concise expression for the additional tidal potential, derived by MacDonald (Rev Geophys 2:467–541, 1964), is combined with a convenient assumption that the quality factor Q is frequency-independent (or, equivalently, that the geometric lag angle is constant in time). This makes the treatment unphysical because MacDonald’s derivation of the said formula was, very implicitly, based on keeping the *time* lag frequency-independent, which is equivalent to setting Q scale as the inverse tidal frequency. This contradiction requires the entire MacDonald treatment of both non-resonant and resonant rotation to be rewritten. The non-resonant case was reconsidered by Efroimsky and Williams (Cel Mech Dyn Astron 104:257–289, 2009), in application to spin modes distant from the major commensurabilities. In the current paper, we continue this work by introducing the necessary alterations into the MacDonald-torque-based model of falling into a 1-to-1 resonance. (The original version of this model was offered by Goldreich (Astron J 71:1–7, 1966). Although the MacDonald torque, both in its original formulation and in its corrected version, is incompatible with realistic rheologies of minerals and mantles, it remains a useful toy model, which enables one to obtain, in some situations, qualitatively meaningful results without resorting to the more rigorous (and complicated) theory of Darwin and Kaula. We first address this simplified model in application to an oblate primary body, with tides raised on it by an orbiting zero-inclination secondary. (Here the role of the tidally-perturbed primary can be played by a satellite, the perturbing secondary being its host planet. A planet may as well be the perturbed primary, its host star acting as the tide-raising secondary). We then extend the model to a triaxial primary body experiencing both a tidal and a permanent-figure torque exerted by an orbiting secondary. We consider the effect of the triaxiality on

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Report Documentation Page			Form Approved OMB No. 0704-0188		
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1. REPORT DATE 2012		2. REPORT TYPE		3. DATES COVERED 00-00-2012 to 00-00-2012	
4. TITLE AND SUBTITLE Bodily tides near the 1:1 spin-orbit resonance: correction to Goldreich's dynamical model				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) US Naval Observatory, Washington, DC, 20392				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited					
13. SUPPLEMENTARY NOTES Celest Mech Dyn Astr (2012)					
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15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT Same as Report (SAR)	18. NUMBER OF PAGES 29	19a. NAME OF RESPONSIBLE PERSON
a. REPORT unclassified	b. ABSTRACT unclassified	c. THIS PAGE unclassified			

both circulating and librating rotation near the synchronous state. Circulating rotation may evolve toward the libration region or toward a spin faster than synchronous (the so-called pseudosynchronous spin). Which behaviour depends on the orbit eccentricity, the triaxial figure of the primary, and the mass ratio of the secondary and primary bodies. The spin evolution will always stall for the oblate case. For libration with a small amplitude, expressions are derived for the libration frequency, damping rate, and average orientation. Importantly, the stability of pseudosynchronous spin hinges upon the dissipation model. Makarove and Efroimsky (Astrophys J, 2012) have found that a more realistic tidal dissipation model than the corrected MacDonald torque makes pseudosynchronous spin unstable. Besides, for a sufficiently large triaxiality, pseudosynchronism is impossible, no matter what dissipation model is used.

Keywords Bodily tides · Land tides · Moon · Libration · Spin-orbit resonance · Planetary satellites

1 Motivation

Bodily tides in a near-spherical homogeneous primary perturbed by a point-like secondary are described by the theory developed mainly by Darwin (1879, 1880) and Kaula (1966). Sometimes this theory is referred to as the *Darwin torque*. Based on a Fourier-like expansions of the perturbing potential and of the tidally-induced potential of the disturbed primary, their theory permits an arbitrary rheology of the primary.

Sometimes a much simpler empirical model, offered by MacDonald (1964) and often named as the *MacDonald torque*, is used in the literature for obtaining very approximate but still qualitatively reasonable description of tidal evolution. This model can be derived from the Darwin–Kaula theory, under several simplifying assumptions. Among these is the assumption that the tidal quality factor Q of the perturbed primary should scale as the inverse of the tidal frequency. Being incompatible with the rheologies of actual minerals, this key assumption prohibits the use of the MacDonald model in long-term orbital calculations.

Despite the unrealistic rheology instilled into the MacDonald model, the model remains a lab which permits one to gain qualitative understanding of tidal evolution (rotational and orbital), without resorting to the lengthy calculations required in the accurate Darwin–Kaula approach. It should be noted however that employment of the MacDonald model needs some care. Historically, the orbital calculations performed with aid of this model by one of its creators, MacDonald (1964), contained an inherent contradiction. MacDonald began his paper with deriving a concise expression for the additional tidal potential, and then combined this expression with a convenient assumption that the geometric lag angle is constant (or, equivalently, that the tidal quality factor is a frequency independent constant). This made the treatment unphysical because MacDonald’s derivation of the said formula was, very implicitly, based on keeping the *time* lag frequency-independent, which is equivalent to setting Q to scale as the inverse tidal frequency. This contradiction made his theory inconsistent. The said oversight has then been repeated many a time in the literature (Kaula 1968, Eq. 4.5.37; Murray and Dermott 1999, Eq. 5.14).¹ In particular, the MacDonald method was used by Goldreich (1966) in his theory of dynamical evolution near the 1:1 spin-orbit reso-

¹ Due to an error in our translation from German, we mis-assumed in our previous papers (Efroimsky and Williams 2009; Efroimsky 2012a; that Gerstenkorn 1955) had based his development on a constant- Q model. Therefore we stated that his theory contained the same genuine inconsistency as the theory by MacDonald (1964). Accurate translation of the work by Gerstenkorn (1955) has shown that his method was based on a constant-time-lag model. Therefore we retract our statement about Gerstenkorn’s approach sharing the

nance. We reconsider this theory, employing the corrected version of the MacDonald torque, i.e., setting the quality factor to scale as inverse frequency.

2 Linear bodily tide in a near-spherical primary

Consider a near-spherical primary of radius R and a secondary of mass M_{sec}^* located at $\vec{r}^* = (r^*, \phi^*, \lambda^*)$, where $r^* \geq R$. The tidal potential created by the secondary alters the primary's shape and, as a result, its potential. For linear tides, the amendment to the primary's exterior potential is known (e.g., [Efroimsky and Williams 2009](#); [Efroimsky 2012a,b](#)) to read as

$$U(\vec{r}) = -GM_{sec}^* \sum_{l=2}^{\infty} k_l \frac{R^{2l+1}}{r^{l+1} r^{*l+1}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi) P_{lm}(\sin \phi^*) \cos m(\lambda - \lambda^*), \quad (1)$$

δ_{ij} being the Kronecker delta, $G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ being Newton's gravity constant, and γ being the angle between the vectors \vec{r}^* and \vec{r} pointing from the primary's centre. As agreed above, $\vec{r}^* = (r^*, \phi^*, \lambda^*)$ denotes the position of the perturber, while $\vec{r} = (r, \phi, \lambda)$ is an exterior point, at a radius $r \geq R$, where the tidal potential amendment $U(\vec{r})$ is measured. The longitudes λ, λ^* are reckoned from a fixed meridian on the primary, while the latitudes ϕ, ϕ^* are reckoned from its equator. Indices l and m are traditionally referred to as the *degree* and *order*, accordingly. The associated Legendre functions $P_{lm}(x)$ (termed associated Legendre *polynomials* when their argument is sine or cosine of some angle) are introduced as in [Kaula \(1966, 1968\)](#), and may be called *unnormalised* associated Legendre functions, to distinguish them from their normalised counterparts.² The Love numbers k_l can be calculated from the geophysical properties of the primary.

A different formula for the tidal-response-generated change in the potential was suggested by [Kaula \(1961, 1964\)](#). Kaula devised a method of switching variables, from the spherical coordinates to the orbital elements $(a^*, e^*, i^*, \Omega^*, \omega^*, \mathcal{M}^*)$ and $(a, e, i, \Omega, \omega, \mathcal{M})$ of the secondaries located at \vec{r}^* and \vec{r} . The goal was to explore how a tide-raising secondary at \vec{r}^* acts on a secondary at \vec{r} through the medium of the tides it creates on their mutual primary.

The development enabled Kaula to process (1) into a series, which was a disguised form of a Fourier expansion of the tide. Interestingly, Kaula himself never referred to that expansion as a Fourier series, nor did he ever write down explicitly the expressions for the Fourier modes. At the same time, the way in which Kaula introduced the phase lags indicates that he was aware of how the modes were expressed via the perturber's orbital elements and the

Footnote 1 continued

inconsistency of MacDonald's theory. We also thank Hauke Hussmann and Peter Noerdlinger for their kind help in translating excerpts from Gerstenkorn's work.

² The Legendre polynomials may be defined through the Rodriguez formula $P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$. In most literature, the associated Legendre functions are introduced, for a nonnegative m , as

$$P_{lm}(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad \text{and} \quad P_l^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x),$$

so that $P_{lm}(x) = (-1)^m P_l^m(x)$. The above definition agrees with the one offered by [Abramowitz and Stegun \(1972, p. 332\)](#). A different convention is accepted in those books (e.g., [Arfken and Weber 1995, p. 623](#)) where $P_l^m(x)$ lacks the $(-1)^m$ factor and thus coincides with $P_{lm}(x)$.

primary's rotation rate. The original works by [Kaula \(1961, 1964\)](#) were written in a terse manner, with many technicalities omitted. A comprehensive elucidation of his approach can be found in the [Efroimsky and Makarov \(2012\)](#). Referring the reader to that paper for details, here we cite only the resulting formula for the secular part of U , in the special case when the tide-raising secondary itself experiences perturbation from the tide it creates on the primary (so $\vec{r}^* = \vec{r}$):

$$U^{(sec)}(\vec{r}) = -\frac{GM_{sec}}{a} \sum_{l=2}^{\infty} \left(\frac{R}{a}\right)^{2l+1} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) \\ \times \sum_{p=0}^l F_{lmp}^2(i) \sum_{q=-\infty}^{\infty} G_{lpq}^2(e) k_l(\omega_{lmpq}) \cos \epsilon_l(\omega_{lmpq}), \quad (2)$$

where l, m, p, q are integers, $F_{lmp}(i)$ are the inclination functions ([Gooding and Wagner 2008](#)), $G_{lpq}(e)$ are the eccentricity polynomials coinciding with the Hansen coefficients $X_{(l-2p+q)}^{(-l-1), (l-2p)}(e)$, while the superscript “sec” means: secular.

The dynamical Love numbers k_l and the phase lags ϵ_l are functions of the Fourier modes

$$\omega_{lmpq} \equiv (l-2p)\dot{\omega} + (l-2p+q)\dot{\mathcal{M}} + m(\dot{\Omega} - \dot{\theta}), \quad (3)$$

θ being the primary's sidereal angle, and $\dot{\theta}$ being its angular velocity. While the tidal modes (3) can be of either sign, the physical forcing frequencies

$$\chi_{lmpq} \equiv |\omega_{lmpq}| = |(l-2p)\dot{\omega} + (l-2p+q)\dot{\mathcal{M}} + m(\dot{\Omega} - \dot{\theta})| \quad (4)$$

at which the stress and strain oscillate in the primary are positive definite.

A partial sum of series (2), with $|l|, |q|, |j| \leq 2$, was offered earlier by [Darwin \(1879\)](#). An explanation of Darwin's method in modern notation can be found in [Ferraz-Mello et al. \(2008\)](#).³

The power of the Darwin–Kaula approach lies in its compatibility with any rheology, i.e., with an arbitrary form of the mode-dependence of the products $k_l \cos \epsilon_l$. It can be demonstrated that, for a homogeneous near-spherical primary, the functional form of the mode dependence of the product $k_l \cos \epsilon_{lmpq}$ is defined by index l solely: $k_l(\omega_{lmpq}) \cos \epsilon_l(\omega_{lmpq})$, the other three indices being attributed to the tidal mode. One can assume that this product depends not on the tidal mode ω_{lmpq} but on the positive definite physical frequency χ_{lmpq} . This however will require some care in derivation of the tidal torque from the above expressions for the potential—see [Efroimsky \(2012a,b\)](#) for details.

We had to write down the secular part of the Kaula expansion of tide, because we shall use it as a benchmark wherewith to compare the empirical expression by [MacDonald \(1964\)](#), which we shall derive below in an accurate manner.

Lacking the ability to accommodate an arbitrary rheology, the MacDonald approach (or the *MacDonald torque*) produces, after a necessary correction, a simple method which can sometimes be employed for getting qualitative understanding of the picture.

3 Tidal torque

Consider a secondary body of mass M_{sec} , located relative to its primary at $\vec{r} = (r, \lambda, \phi)$, where ϕ is the latitude, and λ denotes the longitude reckoned from a meridian fixed on

³ In [Ferraz-Mello et al. \(2008\)](#), the meaning of notations \vec{r} and \vec{r}^* is opposite to ours.

the primary. Let U stand for the tidal-response amendment to the primary's potential. This amendment can be generated either by this secondary itself or by some other secondary of mass M_{sec}^* located at $\vec{r}^* = (r^*, \lambda^*, \phi^*)$. In either case, the primary's tidal response to the gravity of the secondary will render a tidal force and torque acting on the secondary of mass M_{sec} . The torque's component perpendicular to the equator of the primary will be given by:

$$T_z = -M_{sec} \frac{\partial U(\vec{r})}{\partial \lambda}. \quad (5)$$

We would reiterate that (5) is a component of the torque wherewith *the primary acts on the secondary* of mass M_{sec} .

Then its negative will be the appropriate (i.e., orthogonal to the primary's equator) component of the torque wherewith *the secondary acts on the primary*:

$$\mathcal{T}_z(\vec{r}) = -T_z = M_{sec} \frac{\partial U(\vec{r})}{\partial \lambda}. \quad (6)$$

Derivation of formulae (5–6) is presented in Appendix A. These expressions are convenient when the tidal-response potential amendment U is expressed through the spherical coordinates r, λ, ϕ and r^*, λ^*, ϕ^* , as in formula (1).

Whenever the tidal response is expressed as a function of the orbital elements of the secondary and the sidereal angle θ of the primary, it is practical to write down that the perpendicular-to-equator component of the torque acting on the primary as

$$\mathcal{T}_z(\vec{r}) = -M_{sec} \frac{\partial U(\vec{r})}{\partial \theta}, \quad (7)$$

θ standing for the primary's sidereal angle (Efroimsky 2012a,b).

We prefer to employ the terms *primary* and *secondary* rather than *planet* and *satellite*. This choice of terms is dictated by our intention to apply the below-developed machinery to research of tidal dissipation and spin evolution of a satellite. In this setting, the satellite is effectively playing the role of a tidally-distorted primary, its host planet acting as a tide-generating secondary. Whenever the method is applied to exploring the problem of planet despinning, the planet is understood as a primary, the host star being a tide-raising secondary.

4 MacDonald (1964)

When it comes to taking dissipation into account, expression (1) turns out to be far more restrictive than (2), in that (1) becomes applicable to a very specific rheological model. This happens because a straightforward⁴ option of instilling the delay into (1) is to replace in this expression the perturber's coordinates $r^*(t)$, $\phi^*(t)$, $\lambda^*(t)$ with their delayed values, $r^*(t - \Delta t)$, $\phi^*(t - \Delta t)$, $\lambda^*(t - \Delta t)$. For example, instead of $\cos m(\lambda - \lambda^*)$ we should employ

⁴ A more accurate treatment, which cannot be employed within the MacDonald model but is implementable within the Darwin–Kaula approach, is to expand the tide-raising potential W and the tidal-response potential change U into Fourier modes ω_{lmpq} , and then to introduce the Love numbers $k_{lmpq} = k_l(\omega_{lmpq})$, phase lags $\epsilon_{lmpq} = \epsilon_l(\omega_{lmpq})$, and time lags $\Delta t_{lmpq} = \Delta t_l(\omega_{lmpq})$. This formalism (explained in detail in Efroimsky 2012a,b) enables one to express the so-introduced Love numbers through the rheological properties of the primary's mantle, and thereby to model adequately the frequency-dependence of these Love numbers.

An intermediate, purely empirical, option would be to introduce “Love numbers” k_{lm} , as if they were functions of both the degree l and order m . This idea is implemented in the IERS Conventions on the Earth rotation (Petit and Luzum 2010). In the LLR (Lunar Laser Ranging) integration software, tides in the Earth are parameterised by k_{lm} and Δt_{lm} with $l = 2$ and $m = 0, 1, 2$ (Standish and Williams 2012).

$$\cos \left(m \lambda - m \lambda^{*(delayed)} \right) = \cos \left(m \lambda - (m \lambda^* - m \dot{\lambda}^* \Delta t) \right), \quad (8)$$

insertion whereof into (1) yields:

$$U(\vec{r}) = - \sum_{l=2}^{\infty} \frac{k_l G M_{sec}^* R^{2l+1}}{r(t)^{l+1} r^*(t - \Delta t)^{l+1}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi(t)) P_{lm}(\sin \phi^*(t - \Delta t)) \cos m(\lambda - \lambda^* + \dot{\lambda}^* \Delta t). \quad (9)$$

Expressing the longitude via the true anomaly v ,

$$\lambda = -\theta + \Omega + \omega + v + O(i^2) = -\theta + \Omega + \omega + \mathcal{M} + 2e \sin \mathcal{M} + O(e^2) + O(i^2), \quad (10)$$

and neglecting the apsidal and nodal precessions, we obtain:

$$\begin{aligned} & \cos \left((m \lambda - m \lambda^*) + m \dot{\lambda}^* \Delta t \right) \\ &= \cos \left((m \lambda - m \lambda^*) + m (\dot{v}^* - \dot{\theta}^*) \Delta t + O(i^{*2}) \right) \end{aligned} \quad (11)$$

or, in terms of the mean anomaly:

$$\begin{aligned} \cos \left((m \lambda - m \lambda^*) + m \dot{\lambda}^* \Delta t \right) &= \cos \left((m \lambda - m \lambda^*) + m (n^* - \dot{\theta}^*) \Delta t + 2m e^* n^* \Delta t \cos \mathcal{M}^* \right. \\ &\quad \left. + O(e^{*2}) + O(i^{*2}) + O(i^2) \right). \end{aligned} \quad (12)$$

This enables us to write down the potential as

$$\begin{aligned} U(\vec{r}) &= -G M_{sec}^* \sum_{l=2}^{\infty} k_l \frac{R^{2l+1}}{r(t)^{l+1} r^*(t - \Delta t)^{l+1}} \sum_{m=0}^l \frac{(l-m)!}{(l+m)!} (2 - \delta_{0m}) P_{lm}(\sin \phi(t)) \\ &\quad P_{lm}(\sin \phi^*(t - \Delta t)) \cos \left(m(\lambda - \lambda^*) + m(\dot{v}^* - \dot{\theta}^*) \Delta t + O(i^2) + O(i^{*2}) \right), \end{aligned} \quad (13)$$

where $P_{lm}(\sin \phi(t)) P_{lm}(\sin \phi^*(t - \Delta t))$ may be replaced, in the order of $O(i^2) + O(i^{*2}) + O(i i^*)$, with $P_{lm}(0) P_{lm}(0)$. A further simplification can be achieved by taking into account the $l = 2$ contribution only. Omitting the λ -independent term with $m = 0$ (in the expression for the torque, m will become a multiplier after the differentiation of $U(\vec{r})$ with respect to λ), and omitting the $m = 1$ term (as $P_{21}(0) = 0$), we arrive at the expression

$$\begin{aligned} U(\vec{r}) &= -\frac{3}{4} \frac{G M_{sec}^* k_2 R^5}{r(t)^3 r^*(t - \Delta t)^3} \left[1 + O(i^2) + O(i^{*2}) + O(i i^*) \right] \cos \left((2\lambda - 2\lambda^*) \right. \\ &\quad \left. + 2(\dot{v}^* - \dot{\theta}^*) \Delta t + O(i^2) + O(i^{*2}) \right). \end{aligned} \quad (14)$$

In the special case when the tide-raising satellite is the same body as the tidally-perturbed one (i.e., when $r(t) = r^*(t)$, $M_{sec} = M_{sec}^*$, and $\lambda = \lambda^*$), this expression happens to coincide, in the leading order of i , and i^* , with the potential employed by MacDonald (1964). Thus

we have reproduced his empirical approach, by starting with the rigorous formula (1), and by performing the following sequence of approximations:

- First, when accommodating dissipation, we set all the time delays to be equal, for all the physical frequencies $\chi_{lmpq} \equiv |\omega_{lmpq}|$ involved in the tide:

$$\Delta t_{lmpq} \equiv \Delta t(\chi_{lmpq}) = \Delta t. \quad (15)$$

This point is explained in great detail in [Efroimsky and Makarov \(2012\)](#).

- Second, we assume the smallness of the inclinations and lag, through the neglect of the relative errors $O(i^2)$, $O(i^{*2})$, and $O(ii^*)$.
- Third, we truncate the series by leaving only the $l = m = 2$ term.

These three steps take us from (1) to the MacDonald approximation

$$U(\vec{r}) \approx -\frac{3}{4} \frac{GM_{sec}^* k_2 R^5}{r(t)^3 r^*(t - \Delta t)^3} \cos(2\lambda - 2\lambda^* + \epsilon), \quad (16a)$$

where $r^*(t - \Delta t)$ can be replaced,⁵ in the order of $O(e^* n^* \Delta t)$, with $r^*(t)$. With the latter simplification implemented, and with the tide-raising secondary set to coincide with the one perturbed by the tides on the primary (so $r(t) = r^*(t)$, $M_{sec} = M_{sec}^*$, and $\lambda = \lambda^*$), the above expression assumes the form of

$$U(\vec{r}) \approx -\frac{3}{4} GM_{sec} k_2 \frac{R^5}{r^6} \cos \epsilon. \quad (16b)$$

Expressions (14) and (16) contain the *longitudinal lag*

$$\epsilon \equiv m \dot{\lambda}^* \Delta t = 2(\dot{v}^* - \dot{\theta}) \Delta t + O(i^2), \quad (17a)$$

v^* and θ being the true anomaly of the perturber and the sidereal angle of the primary. In the special case (16b), when the tide-generating secondary and the secondary disturbed by the tide on the primary are one and the same body, the asterisks may be dropped:

$$\epsilon \equiv m \dot{\lambda} \Delta t = 2(\dot{v} - \dot{\theta}) \Delta t + O(i^2). \quad (17b)$$

The spin rate $\dot{\theta}$ is a slow variable, in that it may be assumed constant over one orbiting cycle. The true anomaly is a fast variable. For a nonvanishing eccentricity, \dot{v} too is a fast

⁵ From $r = a(1 - e^2)/(1 + e \cos v)$ and $\partial v / \partial M = (1 + e \cos v)^2 / (1 - e^2)^{3/2}$ it is straightforward that

$$\Delta r \equiv r(t) - r(t - \Delta t) = -\frac{ae(1 - e^2)}{(1 + e \cos v)^2} \sin v \Delta v + O(e(\Delta v)^2) = -\frac{ae \sin v}{(1 - e^2)^{1/2}} n \Delta t + O(e(n \Delta t)^2).$$

Thus our replacement of $r^*(t - \Delta t)$ with $r^*(t) = r(t)$ entails a relative error of order $O(en \Delta t)$. In expressions (9), (13), (14), and (16), the absolute error will be of the same order, for $\lambda \neq \lambda^*$. However for $\lambda = \lambda^*$ the absolute error will become $O(en Q^{-1} \Delta t)$, since $\sin \epsilon$ is of the same order as the inverse quality factor Q . In our estimates of errors, it is irrelevant whether we define Q as that appropriate to the principal tidal mode or via formula (19) below; so we simply use the generic notation Q . As we explained in [Efroimsky and Williams \(2009\)](#), after averaging over one revolution of a nonresonant secondary about the primary, the absolute error reduces to $O(e^2 n^2 Q^{-2} (\Delta t)^2)$. In a resonant case, though, we cannot enjoy this reduction, because in this case the averaging procedure looks different as the torque changes its sign over a period of the moon's revolution. For this reason, the absolute error remains $O(en Q^{-1} \Delta t)$.

variable, and so is the lag ϵ . This means that we should take into account these two quantities' variations over an orbital period.

Had we not changed $r^3(t)$ to $r^3(t - \Delta t)$, our expression (16b) would coincide, up to an irrelevant constant, with the leading term of the appropriate formula from MacDonald (1964, Eq. 21). To appreciate this fact, notice that, up to $O(i^2)$, the absolute value of the longitudinal lag ϵ is the double of the geometrical angle $\Delta = |(\dot{\theta} - \dot{\nu}) \Delta t|$ subtended at the primary's centre between the directions to the secondary and to the bulge,⁶ provided Δt is postulated to be the same for all tidal modes.

The analogy between the MacDonald theory and that of Darwin and Kaula can be traced also by starting from the series (2). Consider the case of the tidally perturbed secondary coinciding with the tide-raising one, so $\lambda = \lambda^*$, and all the orbital variables are identical to their counterparts with an asterisk. It will then be easy to notice that, formally (just formally), expression (14) mimics the principal term of the series (2), provided in this term the multiplier G_{200}^2 is replaced with unity, and the principal phase lag $\epsilon_{2200} \equiv 2(n - \dot{\theta}) \Delta t_{2200}$ is replaced with the longitudinal lag (17). This way, within the MacDonald formalism, the longitudinal lag (17) is playing the role of an *instantaneous* phase lag associated with double the *instantaneous* synodic frequency

$$\chi = 2 |\dot{\nu} - \dot{\theta}|, \quad (18)$$

which is, up to $O(i^2)$, the double of the angular velocity wherewith the point located under the secondary (with the same latitude and longitude) is moving over the surface of the primary.

To extend further the analogy between the MacDonald and Darwin–Kaula models, one can *define* an auxiliary quantity

$$“Q” = \frac{1}{\sin |\epsilon|} \quad (19)$$

and derive from (17) and (19) that, in the leading order of ϵ and i , this quantity satisfies

$$“Q” = \frac{1}{\chi \Delta t}. \quad (20a)$$

A popular fallacy would then be to interpret (20a) as a rheological scaling law $Q \sim \chi^\alpha$ with $\alpha = -1$. That this interpretation is generally incorrect follows from the fact that the quantity “ Q ”, *defined* through (19), is not obliged to coincide with the quality factor.⁷

⁶ Be mindful that the double of the geometrical angle is *not* equal to the absolute value of $\epsilon_{2200} = 2(n - \dot{\theta}) \Delta t_{2200}$.

⁷ Within the Darwin–Kaula theory, for each tidal mode ω_{lmpq} , we introduce the phase lag as $\epsilon_{lmpq} \equiv \omega_{lmpq} \Delta t_{lmpq}$. Then we introduce the appropriate quality factor $Q_{lmpq} \equiv Q(\chi_{lmpq}) = Q(|\omega_{lmpq}|)$ via the expression for the one-cycle energy loss:

$$\Delta E_{cycle}(\chi_{lmpq}) = - \frac{2\pi E_{peak}(\chi_{lmpq})}{Q(\chi_{lmpq})}.$$

Finally, using the fact that $\chi_{lmpq} \equiv |\omega_{lmpq}|$ is the frequency of a *sinusoidal* load, we prove that the afore introduced ϵ_{lmpq} and Q_{lmpq} are interconnected as $1/Q = \sin \epsilon$, if E_{peak} denotes the maximal energy, or in a more complex way, if E_{peak} stands for the maximal work (Efroimsky 2012a,b).

Within the MacDonald method, original or corrected as (15), it is not *a priori* clear if the overall peak work (or the overall peak energy stored) and the overall energy loss over a cycle are interconnected via the auxiliary quantity “ Q ” in exactly the same manner as $E_{peak}(\chi_{lmpq})$ and $\Delta E_{cycle}(\chi_{lmpq})$ are interconnected by $Q(\chi_{lmpq})$ in the above expression. (Recall that the total cycle of the tidal load is, generally, nonsinusoidal). Whenever we can prove that

To sidestep these difficulties, it would be safer to write the constant-time-lag rheological law, for small lags, simply as

$$|\epsilon| = \chi \Delta t. \quad (20b)$$

Historically, MacDonald (1964) arrived at his model via empirical reasoning. He certainly realised that the model was applicable to low inclinations only. At the same time, this author failed to notice that the model also implied the frequency-independence of the time lag. In fact, this frequency-independence, (15), is necessary to derive the MacDonald model (16) from the generic expression (1) for the tidal amendment to the primary's potential. This way, equality (15), is *a priori* instilled into the model. In other words, the MacDonald model of tides includes in itself the rheological scaling law (20b) with a constant Δt . Unaware of this circumstance, MacDonald (1964) set the angular lag to be a frequency-independent constant, an assertion equivalent to the time lag scaling as inverse tidal frequency. However, as we just saw above, a consistent derivation of the MacDonald tidal model requires that the time lag be set frequency-independent.⁸ Thus, to be consistent, the MacDonald method must be corrected by applying (15) or, equivalently, (20b). In greater detail, the necessity of this amendment is considered in Efroimsky and Makarov (2012).

Even after the model is combined with rheology (15), predictions of such a theory are of limited use. The problem is that this rheology is radically different from the actual behaviour of solids. As a result, calculations relying on (15) render implausibly long times of tidal despinning—see, for example the discussion and references in Castillo-Rogez et al. (2011). This tells us that in realistic settings the MacDonald-style approach (14) based on (15) is inferior, compared to the Darwin and Kaula method (2), which may, in principle, be applied to any rheology. Despite this, the MacDonald torque remains a convenient toy model, capable of furnishing results which are qualitatively acceptable over not too long timescales (Hut 1981; Dobrovolskis 2007).

5 The MacDonald torque and the ensuing dynamical model by Goldreich: the case of an oblate body

In this section, we shall trace how the MacDonald theory of bodily tides yields the model of near-resonance spin dynamics by Goldreich (1966). Then we shall recall an oversight in the MacDonald theory, and shall demonstrate that correction of that oversight brings a minor alteration into Goldreich's model of spin evolution.

Footnote 7 continued

$$\Delta E_{\text{cycle}}^{(\text{overall})} = - \frac{2 \pi E_{\text{peak}}^{(\text{overall})}}{“Q”},$$

our “*Q*” can be spared of the quotation marks and can be called *instantaneous* quality factor, while (20a) can be treated as a reasonable approximation to scaling law $Q \sim \chi^\alpha$ with $\alpha = -1$. However, this should be justified in each particular case, not taken for granted.

⁸ In application to a non-resonant setting, a constant- Δt approach was taken yet by Darwin (1879). Later, MacDonald (1964) abandoned this method in favour of a constant-geometric lag calculation. Soon afterwards, though, Singer (1968) advocated for reinstatement of the constant- Δt method. Doing so, he was motivated by an apparent paradox in MacDonald's treatment. As explained by Efroimsky and Williams (2009), the paradox is nonexistent. Nonetheless, the work by Singer (1968) was fruitful at the time, as it renewed the interest in the constant- Δt treatment. The full might of the model was revealed by Mignard (1979, 1980, 1981) who used it to develop closed expressions for the tidal force and torque.

The goal of this section is limited to consideration of the tidal torque solely. So we assume that the body is oblate, and the triaxiality-caused torque does not show up. Appropriate for spinning gaseous objects, this treatment indicates that the tidal torque stays finite for synchronous rotation and vanishes at an angular velocity slightly faster than synchronous.

In Sect. 6 below we shall address the more general setting appropriate to telluric bodies, with triaxiality included. While some authors (e.g., Heller et al. 2011) ignore the triaxiality-caused torque in their treatment of solid planets, it turns out that inclusion of the permanent-figure torque renders important physical consequences and changes the picture completely, as will be seen in Sect. 6.

5.1 The MacDonald torque

We shall restrict ourselves to the case of the tidally perturbed secondary coinciding with the tide-raising one, so $M_{sec} = M_{sec}^*$, and all the orbital variables are identical to their counterparts with an asterisk.

Differentiating (14) with respect to λ , and then setting $\lambda = \lambda^*$, we obtain the following expression for the polar component of the torque, for low i :

$$\begin{aligned} \mathcal{T}_z &= \frac{3}{2} G M_{sec}^2 k_2 \frac{R^5}{r^3(t) r^{*3}(t - \Delta t)} \sin(2(\dot{\nu} - \dot{\theta}) \Delta t) + O(i^2/Q) \\ &= \frac{3}{2} G M_{sec}^2 k_2 \frac{R^5}{r^6} \sin(2(\dot{\nu} - \dot{\theta}) \Delta t) + O(i^2/Q) + O(en\Delta t/Q), \end{aligned} \quad (21)$$

where the error $O(en\Delta t/Q)$ emerges when we identify the lagging distance $r^*(t - \Delta t)$ with $r^*(t) = r(t)$, as explained in footnote 5 in the preceding section. Referring the Reader to Efroimsky and Williams (2009) for this and other technicalities, we would mention that (21) is equivalent to the Darwin torque only under the condition that the rheological model (15–20a) is accepted. For potentials, employment of model (15–20a) enables one to wrap up the infinite series (2) into the elegant finite form (9). For torques (truncated to $l = 2$ only), similar wrapping of the appropriate series is available within the said model.

In the preceding section, we explained the geometric meaning of the longitudinal lag (17): its absolute value is the double of the geometric angle separating the directions to the bulge and the secondary as seen from the primary's centre.

If we define a quantity “ Q ” via (20a), the MacDonald torque will look:

$$\begin{aligned} \mathcal{T}_z &= \frac{3}{2} G M_{sec}^2 k_2 \frac{R^5}{r^3(t) r^{*3}(t - \Delta t)} \sin \epsilon + O(i^2/Q) \\ &= \frac{3}{2} G M_{sec}^2 k_2 \frac{R^5}{r^6} \frac{1}{\text{“}Q\text{”}} \operatorname{sgn}(\dot{\nu} - \dot{\theta}) + O(i^2/Q) + O(en\Delta t/Q). \end{aligned} \quad (22)$$

For a nonzero eccentricity, the quantity “ Q ” should not be interpreted as an instantaneous quality factor, because it is *not* guaranteed to interconnect the peak work or peak energy and the one-cycle energy loss in a manner appropriate to a quality factor—see footnote 7 in the previous section. Therefore a more reasonable and practical way of writing the MacDonald torque (21) would be through using (20b) or (17):

$$\mathcal{T}_z = \frac{3}{2} G M_{\text{sec}}^2 k_2 \frac{R^5}{r^6} \Delta t \, 2 (\dot{\nu} - \dot{\theta}) + O(i^2/Q) + O(en\Delta t/Q). \quad (23)$$

As we saw above, the MacDonald model is self-consistent only for Δt being a frequency-independent constant. However the factor $\dot{\nu} - \dot{\theta}$ showing up in (23) varies over a cycle, for which reason the torque needs averaging. This averaging is carried out in Appendix B. In the vicinity of the 1:1 resonance (for $\dot{\theta}$ close to n), the sign of $\dot{\nu} - \dot{\theta}$ changes twice over a cycle, which makes the averaging procedure nontrivial. This situation is to be addressed in Sects. 5.2 and 5.3.

5.2 Goldreich (1966): treatment based on the MacDonald torque

In this subsection, we shall briefly recall a presently conventional method pioneered almost half a century ago by Goldreich (1966). Based on the MacDonald tide theory, this method has inherited both its simplicity and its flaws.

The 1:1 resonance takes place when the spin rate $\dot{\theta}$ of the primary (the satellite) is equal to the mean motion n wherewith the secondary body (the planet) is apparently orbiting the primary. The formula (22) for the MacDonald torque contains not the difference $\dot{\theta} - n$ but the difference $\dot{\theta} - \dot{\nu}$, for which reason the expression under the integral may twice change its sign in the course of one revolution.

With aid of the formula

$$\nu = \mathcal{M} + 2e \sin \mathcal{M} + \frac{5}{4} e^2 \sin 2\mathcal{M} + O(e^3) \quad (24)$$

and under the assumption that $\dot{e} \ll n$, we shape the difference of our concern into the form of

$$\begin{aligned} \dot{\theta} - \dot{\nu} &= (\dot{\theta} - n) - 2ne \left(\cos \mathcal{M} + \frac{5}{4} e \cos 2\mathcal{M} \right) + O(e^3) \\ &= \dot{\eta} - 2ne \left(\cos \mathcal{M} + \frac{5}{4} e \cos 2\mathcal{M} \right) + O(e^3) \end{aligned} \quad (25)$$

or, equivalently,

$$\frac{\dot{\theta} - \dot{\nu}}{2ne} = - \left[\cos \mathcal{M} + \frac{5}{4} e \cos 2\mathcal{M} - \frac{\dot{\eta}}{2ne} + O(e^2) \right]. \quad (26)$$

Here

$$\eta \equiv \theta - \mathcal{M} - \omega - \Omega \quad (27)$$

is a slowly changing quantity, whose time-derivative $\dot{\eta} \equiv \dot{\theta} - n$ becomes nil when the system goes through the resonance.⁹ To impart the words “slowly changing” with a definite meaning, we assert that $\dot{\eta}/n$ is of order e^2 —a claim to be justified *a posteriori*.

Expression (25) changes its sign at the points where

$$\cos \mathcal{M} = -\frac{5}{4} e \cos 2\mathcal{M} + \frac{\dot{\eta}}{2ne} + O(e^2). \quad (28)$$

⁹ Goldreich (1966) defined this quantity simply as $\eta \equiv \theta - \mathcal{M}$, because he reckoned θ from a fixed perihelion direction. We however reference our θ from a direction fixed in space. (It is the same direction wherefrom the node is reckoned). This convention originates from our definition of θ as the sidereal angle—it is in this capacity that θ was introduced back in Eqs. (3–4). As we are not considering the nodal or apsidal precession, our subsequent formulae containing $\dot{\theta}$ will be equivalent to those ensuing from Goldreich’s definition of θ .

As all the terms on the right-hand side of (28) are of order e at most, so must be the term on the left-hand side. Hence condition (28) is obeyed in the two points whose mean anomaly (and therefore also true anomaly) is close to $\pm \pi/2$:

$$\nu = \pm \left(\frac{\pi}{2} - \delta \right), \quad (29)$$

δ being of order e . From (29) and (24) we obtain:

$$\begin{aligned} \sin \delta &= \cos \nu = \cos (\mathcal{M} + 2e \sin \mathcal{M} + O(e^2)) \\ &= \cos \mathcal{M} - 2e \sin^2 \mathcal{M} + O(e^2) \end{aligned} \quad (30)$$

which, in combination with (28), entails:

$$\begin{aligned} \sin \delta &= -\frac{5}{4}e \cos 2\mathcal{M} + \frac{\dot{\eta}}{2ne} - 2e \sin^2 \mathcal{M} + O(e^2) \\ &= \frac{\dot{\eta}}{2ne} - e \left(\frac{5}{4} - \frac{1}{2} \sin^2 \mathcal{M} \right) + O(e^2). \end{aligned} \quad (31)$$

Insertion of (29) into (24) also yields $\pm \sin \mathcal{M} = \cos \delta + O(e)$. Recalling that δ is of order e , we obtain: $\sin^2 \mathcal{M} = 1 - \sin^2 \delta + O(e) = 1 + O(e)$. This enables us to rewrite (31) as

$$\delta = \frac{\dot{\eta}}{2ne} - \frac{3}{4}e + O(e^2). \quad (32)$$

Before finding the rate $\dot{\eta} \equiv \dot{\theta} - n$ at which the resonance is traversed, let us enquire if perhaps it could be simply put nil, the satellite being permanently kept in the resonance. The answer is negative, because for a vanishing $\dot{\theta} - n$ the difference $\dot{\theta} - \dot{\nu}$ emerging in (25) becomes a varying quantity of an alternating sign, and so becomes the torque. On general grounds, one should not expect that the average torque becomes nil for a vanishing $\dot{\eta}$, though it may vanish for some finite value of $\dot{\eta}$.

To find this value of $\dot{\eta}$, many authors (Goldreich 1966, Eq. 15; Kaula 1968, Eq. 4.5.29; Murray and Dermott 1999, Eq. 5.11) simply integrated the MacDonald torque (22), assuming the quality factor Q constant, and thus keeping it outside the integral:

$$\begin{aligned} \langle T_z \rangle^{(\text{Goldreich})} &= -\frac{3GM_{\text{sec}}^2 k_2 R}{4\pi a^2 Q} \frac{1}{(1-e^2)^{1/2}} \int_0^{2\pi} \frac{R^4}{r^4} \operatorname{sgn}(\dot{\theta} - \dot{\nu}) d\nu \\ &= -\frac{3GM_{\text{sec}}^2 k_2 R}{4\pi a^2 Q} \frac{2}{(1-e^2)^{1/2}} \left[\int_0^{\pi/2-\delta} \frac{R^4}{r^4} d\nu - \int_{\pi/2-\delta}^{\pi} \frac{R^4}{r^4} d\nu \right] \\ &= \frac{3GM_{\text{sec}}^2 k_2 R^5}{2\pi a^6 Q} \left[\int_0^{\pi/2-\delta} - \int_{\pi/2-\delta}^{\pi} \right] (1 + 4e \cos \nu) d\nu + O(e^2) \end{aligned} \quad (33)$$

$$= \frac{3GM_{\text{sec}}^2 k_2 R^5}{\pi a^6 Q} (4e \cos \delta - \delta) + O(e^2). \quad (34)$$

In a trapping situation, the average torque vanishes. So (34) entails:

$$\delta = 4e \cos \delta = 4e + O(e^2). \quad (35)$$

By combining the latter with (32), the afore quoted authors arrived at

$$\dot{\eta}_{\text{stall}} = \frac{19}{2} n e^2, \quad (36)$$

an expression repeated later in papers and textbooks (e.g., Eq. 4.5.37 in Kaula 1968, or Eq. 5.14 in Murray and Dermott 1999). This result however needs to be corrected, because the MacDonald approach is incompatible with the frequency-independence of Q assumed in (34).

Be mindful that (36), as well as its corrected version (44) to be derived below, indicate that $\dot{\eta}$ is of order e^2 . This justifies our assertion made in the paragraph after formula (27).

5.3 The corrected MacDonald model

To impart the MacDonald treatment consistently, one has to calculate the averaged torque, with the frequency-dependence of Q taken into consideration. As we saw in Sect. 4, the MacDonald approach fixes this dependence in a manner that can, with some reservations, be approximated with

$$Q = \frac{1}{\chi \Delta t} \quad (37)$$

or, in more general notations,

$$Q = \mathcal{E}^\alpha \chi^\alpha, \quad \text{with } \alpha = -1, \quad (38)$$

where the double instantaneous synodic frequency is given by (18).

The form (38) of the rheological law is more convenient, as it leaves us an opportunity to consider values of α different from -1 . For any value of α , the constant \mathcal{E} is an integral rheological parameter, which has the dimension of time, and whose physical meaning is discussed in Efroimsky and Lainey (2007). It can be demonstrated that in the special case of $\alpha = -1$ the parameter \mathcal{E} coincides with the time lag Δt . For actual terrestrial bodies, α is different from -1 , and the integral rheological parameter \mathcal{E} is related to the time lag in a more complicated manner (*Ibid.*).

As demonstrated in Appendix B, insertion of (38) and (18) into (22), for $\alpha = -1$, or equivalently, direct employment of (23), entails the following expression for the orbit-averaged torque acting on a secondary:

$$\langle \mathcal{T}_z \rangle = -\mathcal{Z} \left[\dot{\theta} \mathcal{A}(e) - n \mathcal{N}(e) \right] + O(i^2/Q) + O(Q^{-3}) + O(en\Delta t/Q), \quad (39)$$

where

$$\mathcal{A}(e) = \left(1 + 3e^2 + \frac{3}{8}e^4 \right) (1 - e^2)^{-9/2} = 1 + \frac{15}{2}e^2 + \frac{105}{4}e^4 + O(e^6) \quad (40)$$

and

$$\begin{aligned} \mathcal{N}(e) = & \left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6 \right) (1 - e^2)^{-6} = 1 + \frac{27}{2}e^2 + \frac{573}{8}e^4 \\ & + O(e^6), \end{aligned} \quad (41)$$

while the factor \mathcal{Z} is given by¹⁰

¹⁰ Recall that, for $\alpha = -1$, the rheological parameter \mathcal{E} is simply the time lag Δt .

$$\mathcal{Z} = \frac{3 G M_{sec}^2 k_2 \mathcal{E}}{R} \frac{R^6}{a^6} = \frac{3 n^2 M_{sec}^2 k_2 \Delta t}{(M_{prim} + M_{sec})} \frac{R^5}{a^3} = \frac{3 n M_{sec}^2 k_2}{Q (M_{prim} + M_{sec})} \frac{R^5}{a^3} \frac{n}{\chi}, \quad (42)$$

M_{prim} and M_{sec} being the masses of the primary and the secondary.

Be mindful that the right-hand side of (42) contains a multiplier $\frac{n}{\chi} = \frac{n}{2|\dot{\theta} - \dot{\nu}|}$ which is missing in the despinning formula employed by Correia and Laskar (2004, 2009). This happened because in *Ibid.* the quality factor was introduced as $1/(n \Delta t)$ and not as $1/(\chi \Delta t)$ —see the line after formula (9) in Correia and Laskar (2009). In reality, the quality factor Q must, of course, be a function of the forcing frequency χ (which happens to coincide with the mean motion n in the 3:2 and 1:2 resonances but differs from n outside these).

The quality factor Q being (within this model) inversely proportional to χ , the presence of the $\frac{n}{\chi}$ factor in (42) makes the overall factor \mathcal{Z} a frequency-independent constant.

Rewriting (39) as

$$\begin{aligned} \langle \mathcal{T}_z \rangle &= -\mathcal{Z} \left[\dot{\theta} \left(1 + \frac{15}{2} e^2 + \frac{105}{4} e^4 \right) - n \left(1 + \frac{27}{2} e^2 + \frac{573}{8} e^4 \right) \right] + O(e^6) \\ &\quad + O(i^2/Q) + O(en\Delta t/Q) \\ &= -\mathcal{Z} \left[\dot{\eta} \left(1 + \frac{15}{2} e^2 + \frac{105}{4} e^4 \right) - 6ne^2 \left(1 + \frac{121}{16} e^2 \right) \right] + O(e^6) \\ &\quad + O(i^2/Q) + O(en\Delta t/Q), \end{aligned} \quad (43)$$

we see that it vanishes when the rate of change of $\eta = \theta - \mathcal{M} - \omega - \Omega$ accepts the value

$$\dot{\eta}_{stall} = 6 n e^2 + \frac{3}{8} n e^4 + O(e^6) + O(i^2/Q) + O(en\Delta t/Q). \quad (44)$$

For $\dot{\eta}$ larger or smaller than $\dot{\eta}_{stall}$, the average torque (43) is nonzero and impels $\dot{\eta}$ to evolve towards the stall value (44).

On the right-hand side of (44), the leading-order term contains a numerical factor of 6, as different from the factor 19/2 showing up in (36). The necessity to change 19/2 to 6 in the e^2 term was pointed out by Rodríguez et al. (2008, Eq. 2.4), who had arrived at this conclusion through some different considerations [which, too, were based on the frequency-dependence (38)]. The same result can be obtained within the Darwin–Kaula approach, provided the scaling law (38) is employed (Efroimsky 2012a,b).¹¹

Although correction of the oversight in the MacDonald torque renders a number different from the one furnished by Goldreich’s development (6 instead of 19/2), qualitatively the principal conclusion by Goldreich (1966) remains unchanged: when an oblate body’s spin is evolving toward the resonance, vanishing of the average tidal torque entails spin slightly faster than resonant, a so-called *pseudosynchronous rotation*. It however should be strongly emphasised that the possibility of pseudosynchronous rotation hinges upon the dissipation model employed. Makarov and Efroimsky (2012) have found that a more realistic tidal dissipation model than the corrected MacDonald torque makes pseudosynchronous rotation impossible.

¹¹ Implicitly, this result is present also in Correia et al. (2011, eqn 20), Laskar and Correia (2004), eqn 9, and in Hut (1981). The earliest implicit occurrence of this result was in Goldreich and Peale (1966, eqn 24).

6 Evolution of rotation near the 1:1 resonance: the case of a triaxial body

In distinction from a gaseous or liquid body, a solid body would be expected to have a permanent figure in addition to the tidal distortion discussed so far. Goldreich (1966) demonstrated that this permanent figure plays an important role in determining if a primary, whose rotation is being slowed down or sped up by the tidal torque caused by a secondary, can be captured into the synchronous rotation state. This section follows Goldreich's derivation, but substitutes the MacDonald tidal torque with its corrected version, and also uses a more general mass expression.

6.1 Rotating primary subject to a triaxiality-caused torque: the equation of motion and the first integral

Consider a triaxial primary body, which has its principal moments of inertia ordered as $A < B < C$, and which is rotating about the maximal-inertia axis associated with moment C . About the primary, a secondary body describes a near-equatorial orbit (so its inclination on the primary's equator, i , may be neglected). The secondary exerts on the primary two torques. One being tidal, the other is triaxiality-caused, i.e., generated by the existence of the permanent figure of the primary. Its component acting on the primary about the maximal-inertia axis is

$$\mathcal{T}_{\text{triax}} = \frac{3}{2} (B - A) \frac{GM_{\text{sec}}}{r^3} \sin 2\lambda, \quad (45)$$

the longitude λ being furnished by formula (10). In that formula, the sidereal angle θ is reckoned from a reference direction in space to the principal axis associated with moment A .

The acceleration of the sidereal angle then obeys

$$C \ddot{\theta} - \frac{3}{2} (B - A) \frac{GM_{\text{sec}}}{r^3} \sin 2\lambda = 0. \quad (46)$$

In neglect of the nodal and apsidal precession, as well as of $\dot{\mathcal{M}}_0$, definition (27) yields:¹²

$$\dot{\eta} = \dot{\theta} - n. \quad (47)$$

Ignoring changes in the mean motion,¹³ we write:

$$\ddot{\theta} = \ddot{\eta}, \quad (48)$$

so the first term in (46) becomes simply $C \ddot{\eta}$.

¹² The caveat about three neglected items implies that our n is the *osculating* mean motion $n(t) \equiv \sqrt{\mu/a(t)^3}$, and that we extend this definition to perturbed settings. The so-defined mean anomaly evolves in time as $\mathcal{M} = \mathcal{M}_0(t) + \int_0^t n(t) dt$, whence $\dot{\mathcal{M}} = \dot{\mathcal{M}}_0 + n(t)$.

The said caveat becomes redundant when n is defined as the *apparent* mean motion, i.e., either as the mean-anomaly rate $d\mathcal{M}/dt$ or as the mean-longitude rate $dL/dt = d\Omega/dt + d\omega/dt + d\mathcal{M}/dt$ (Williams et al. 2001). While the first-order perturbations of $a(t)$ and of the osculating mean motion $\sqrt{\mu/a(t)^3}$ include no secular terms, such terms are often contained in the epoch terms $\dot{\Omega}$, $\dot{\omega}$, and $\dot{\mathcal{M}}_0$. This produces the difference between the apparent mean motion defined as dL/dt (or as $d\mathcal{M}/dt$) and the osculating mean motion $\sqrt{\mu/a(t)^3}$.

¹³ Our neglect of evolution of the mean motion is acceptable, because orbital acceleration \dot{n} is normally much smaller than the spin acceleration $\ddot{\theta}$. Indeed, for torques arising from the primary, $\dot{n}/\ddot{\theta}$ is of the same order as the ratio of C to the orbital moment of inertia.

To process the second term in (46), we would compare (10) with (27):

$$\begin{aligned}\lambda &= -\theta + \Omega + \omega + \nu + O(i^2) = (-\theta + \Omega + \omega + \mathcal{M}) + (\nu - \mathcal{M}) + O(i^2) \\ &= -\eta + (\nu - \mathcal{M}) + O(i^2)\end{aligned}\quad (49)$$

In neglect of the inclination, the following approximation is acceptable in the vicinity of the 1:1 resonance:¹⁴

$$\langle r^{-3} \sin 2\lambda \rangle = -G_{200}(e) a^{-3} \sin 2\eta, \quad (50)$$

where $\langle \cdots \rangle$ signifies *orbital* averaging, while the eccentricity function can be approximated with

$$G_{200}(e) = 1 - \frac{5}{2} e^2 + O(e^4). \quad (51)$$

This way, omitting $O(i^2)$ in (49) and substituting $\sin 2\lambda$ with its average in (46), we transform the second term in (46) to:

$$+ \frac{3}{2} (B - A) \frac{GM_{sec}}{a^3} G_{200}(e) \sin 2\eta. \quad (52)$$

While Goldreich (1966) assumed that the mass of the secondary is much larger than the mass of the primary, we do not impose this restriction. Combining Kepler's third law, $G(M_{sec} + M_{prim})/a^3 = n^2$, with formulae (46), (48), and (52), we finally arrive at

$$C\ddot{\eta} + \frac{3}{2} (B - A) \frac{M_{sec}}{M_{sec} + M_{prim}} n^2 G_{200}(e) \sin 2\eta = 0, \quad (53)$$

an equation describing the evolution of the rotation angle η . As pointed out by Goldreich (1966), this equation is equivalent to the one describing a simple pendulum. Indeed, in terms of $\beta = 2\eta$, Eq. (53) becomes $\ddot{\beta} + \chi_{lib-max}^2 \sin \beta = 0$, with a constant positive $\chi_{lib-max}^2$ and with β playing the role of the pendulum angle.¹⁵

It follows directly from (53) that the spin acceleration $\ddot{\eta}$ vanishes if η assumes the values of 0 or π . As can be seen from the pendulum analogy, the initial conditions $(\eta, \dot{\eta})_{t=t_0} = (0, 0)$, as well as the conditions $(\eta, \dot{\eta})_{t=t_0} = (\pi, 0)$, correspond to the situation where the pendulum comes to a stall in the lower point (so $\dot{\beta} = 0$ when $\beta = 0$ or 2π). Under such initial conditions, η stays 0 or π all the time, which implies a uniform synchronous rotation of the secondary about the primary.¹⁶

Multiplication of Eq. (53) by $\dot{\eta}$, with subsequent integration over time t , gives the first integral of motion,

$$\frac{1}{2} C \dot{\eta}^2 - \frac{3}{4} (B - A) \frac{M_{sec}}{M_{sec} + M_{prim}} n^2 G_{200}(e) \cos 2\eta = E, \quad (54)$$

whose value depends on the initial conditions.

¹⁴ Pioneered by Goldreich and Peale (1966), the approximation is explained in more detail by Murray and Dermott (1999) whose treatment omits terms of the order of e^3 and higher (see formulae 5.59–5.60 in *Ibid.*). To justify the omission, recall that before averaging the expansion of $r^{-3} \sin 2\lambda$ includes a series of terms with different arguments of the form $\sin(2\eta + q\mathcal{M})$, where q is an integer. Since we are interested in dynamics in the vicinity of the 1:1 resonance, where η is a slow variable, then only the $\sin(2\eta)$ term remains after the orbital averaging. The averaged-out terms are of the order of e and higher powers including the e^3 terms.

¹⁵ In Sect. 6.5 below, we write down the expression for the libration frequency $\chi_{lib-max}$ and also explain the reason why we equip it with such a subscript—see formulae (68) and (69).

¹⁶ According to (53), the spin acceleration $\ddot{\eta}$ vanishes also for $\eta = \pm\pi/2$, which is the upper point of the pendulum.

The period of the variable η is given by a quadrupled integral over a quarter-cycle of η :

$$P = 4 \int_{\eta=0}^{\eta=\eta_{max}} \frac{d\eta}{\dot{\eta}}. \quad (55)$$

For circulation, use $\eta_{max} = \pi/2$. In the case of libration about $\eta = 0$, the value of η_{max} is less than $\pi/2$ and can be expressed via E by setting $\dot{\eta} = 0$ in Eq. (54). To get an explicit expression of P , one should first express $\dot{\eta}$ via E using (54) and taking the positive root, and then should plug the so-obtained expression for $\dot{\eta}$ into (55). Also be mindful that libration about π can be converted to the same integral by adding or subtracting π from the variable of integration.

In what follows, averaging over the period P will be denoted by $\langle \cdots \rangle_P$, the subscript serving to distinguish the operation from averaging over the orbit employed in Sect. 5 and in formula (50).

6.2 Parallels with pendulum: the auxiliary quantity W

Goldreich (1966) noted the similarity of the differential equations (53–54) to the classical pendulum problem. If the initial conditions place the body outside of the 1:1 spin-orbit resonance, then η circulates with a forced oscillation in rotation that depends on the mean value $\langle \dot{\eta} \rangle_P$ of the $\dot{\eta}$ frequency. If however the initial conditions place the body's spin within a sufficiently close proximity of the 1:1 resonance (the “resonance region”), then η will librate about 0 or π . This will be a free physical libration with an amplitude smaller than $\pi/2$. Inside the resonance region, the initial conditions establish the free-libration amplitude and phase. However the restricted nature of the motion will keep the mean value of η constant: it will be either 0 or π . For the same reason, $\langle \dot{\eta} \rangle_P$ will remain zero in the libration regime. In contrast to this, outside of the resonance region the mean rate of circulation $\langle \dot{\eta} \rangle_P$ will possess a value determined by the initial conditions on η and $\dot{\eta}$.

Employing (54) to express $\dot{\eta}$ via E , and plugging this expression into (55), Goldreich (1966) demonstrated that the period P can be expressed via E , inside or outside the resonance region, by a complete elliptic integral of the first kind. This is natural, as the expression of $\dot{\eta}$ through E involves a square root.

Just as in the pendulum case, there exists a critical E for which the period diverges. This is the boundary $E = E_b$ separating circulation from libration. To locate the boundary, one should set simultaneously $\eta = \pi/2$ and $\dot{\eta} = 0$ in formula (54):

$$E_b = \frac{3}{4} (B - A) \frac{M_{sec}}{M_{sec} + M_{prim}} G_{200}(e) n^2. \quad (56)$$

Mathematically, the emergence of a logarithmic singularity in P at $E = E_b$ can be observed from formulae (9–10) in *Ibid.* Physically, this situation resembles the slowing-down of a pendulum at the circulation/libration border. In our problem, though, this division corresponds to $\eta = \pm\pi/2$.

By setting simultaneously $\eta = 0$ and $\dot{\eta} = 0$ in (54), we obtain the minimal value that the integral E can assume:

$$\min E = -E_b. \quad (57)$$

The values $E > E_b$ correspond to circulation in either direction, with oscillations of the rotation rate. The values falling within the interval $E_b > E > -E_b$ give libration. The minimum value $E = -E_b$ corresponds to synchronous rotation without libration.

The quantity η being slowly varying (compared to the mean motion n), the period P , with which η is changing, naturally turns out to be much longer than the orbital period, both outside and inside the 1:1 resonance:

$$P \gg \frac{2\pi}{n}. \quad (58)$$

This can be appreciated from the evident equalities

$$P = \frac{2\pi}{|\langle \dot{\eta} \rangle_P|} \quad \text{outside of the 1:1 resonance,} \quad (59a)$$

$$P = \frac{2\pi}{\chi_{lib}} \quad \text{inside the 1:1 resonance.} \quad (59b)$$

In (59b), χ_{lib} denotes the libration frequency which too is much smaller than the mean motion n , as we shall see shortly.

A useful quantity defined by Goldreich (1966) was

$$W \equiv P \langle \dot{\eta}^2 \rangle_P \equiv \int_0^P \dot{\eta}^2 dt = 4 \int_{\eta=0}^{\eta=\eta_{max}} \dot{\eta} d\eta. \quad (60)$$

Similarly to (55), $\eta_{max} = \pi/2$ for circulation, and $\eta_{max} < \pi/2$ for libration. Just as P , so W can be expressed through E by complete elliptic integrals. For circulating $\langle \dot{\eta} \rangle_P$, Goldreich described P and W as applying to one oscillation, but they describe one circulation of η with two oscillations. For librations, P and W describe one libration cycle.

One more quantity of use in this problem will be the mean square variation of $\dot{\eta}$ about $\langle \dot{\eta} \rangle_P$, given by $\langle \dot{\eta}^2 \rangle_P - \langle \dot{\eta} \rangle_P^2$. Be mindful that the notation $\langle \cdots \rangle_P$ is employed to indicate averaging over a circulation cycle as well as that over a libration cycle.

When the values of the mean motion n , the mass factor $M_{sec}/(M_{sec} + M_{prim})$, and the ratio $(B - A)/C$ are given, and the initial conditions on η and $\dot{\eta}$ are set, Eq. (54) furnishes the value of E/C . The comparison of this value with E_b/C distinguishes circulation from libration. With the values of $(B - A)/C$ and E/C known, the complete elliptic integrals can be evaluated and the values of the quantities P , W , $\langle \dot{\eta} \rangle_P$, and $\langle \dot{\eta}^2 \rangle_P$ can be found. It should also be possible to reverse this procedure. For example, the knowledge of the circulating value of $\langle \dot{\eta} \rangle_P$, along with the knowledge of n , the mass factor $M_{sec}/(M_{sec} + M_{prim})$, and the ratio $(B - A)/C$, should allow E/C , W , and $\langle \dot{\eta}^2 \rangle_P$ to be computed.

6.3 The tidal torque and the libration bias

To account for tidal dissipation, Goldreich (1966) added the averaged (over an orbital period) tidal torque (34) to the right-hand side of Eq. (53). We however shall employ the corrected average torque (43) instead of (34). This will result in

$$\begin{aligned} C\ddot{\eta} + \frac{3}{2}(B - A) \frac{M_{sec}}{M_{sec} + M_{prim}} n^2 G_{200}(e) \sin 2\eta \\ = -\mathcal{Z} \left[\dot{\eta} \left(1 + \frac{15}{2} e^2 \right) - 6n e^2 \left(1 + \frac{121}{16} e^2 \right) \right] + O(e^6) + O(i^2/Q) \\ + O(en\Delta t/Q), \end{aligned} \quad (61)$$

where the coefficient \mathcal{Z} depends via (42) on the orbital variables, the tidal parameters, and the masses.

As we already mentioned at the beginning of Sect. 5, ignoring the permanent-figure term would lead one to the conclusions that the tidal torque is finite for synchronous rotation and that it vanishes for a spin rate slightly higher than synchronous. However inclusion of the permanent-figure term into the picture alters the results radically. The constant term on the right-hand side of Eq. (61) causes the mean value of η to be slightly larger than 0 or π . For small librations, this bias is

$$\eta_{bias} = \frac{2 \mathcal{Z} e^2}{(B - A)n} \frac{M_{sec} + M_{prim}}{M_{sec}} \left(1 + \frac{121}{16} e^2 \right) \frac{1}{G_{200}(e)} + O(e^6) \\ + O(i^2/Q) + O(en\Delta t/Q) \quad (62a)$$

$$= \frac{2 \mathcal{Z} e^2}{(B - A)n} \frac{M_{sec} + M_{prim}}{M_{sec}} \left(1 + \frac{161}{16} e^2 \right) + O(e^6) \\ + O(i^2/Q) + O(en\Delta t/Q) \quad (62b)$$

with \mathcal{Z} calculated through (42). The value of the time lag Δt entering the expression (42) for \mathcal{Z} should be set appropriate to the orbital frequency (mean motion).

As evident from (62), the bias comes into being due to the eccentric shape of the orbit. Bodies with permanent figures can achieve synchronous rotation because the bias in η gives birth to a permanent-figure torque that balances the constant dissipative torque.

At this point, an important caveat will be in order. As we explained in Sect. 4, the MacDonald model of tides becomes self-consistent only when the time delay Δt emerging in (42) is set frequency-independent. This circumstance limits the precision of the MacDonald torque and of the Goldreich dynamical model based thereon, whenever the model gets employed to determine the timescales of spin evolution (Efroimsky and Lainey 2007). Nevertheless, for very slow evolution the model may be employed for obtaining qualitative estimates. In this case, Δt should be treated as a parameter that itself depends upon the forcing frequency in the material. For rotation outside the 1:1 spin-orbit lock, it would be a tolerable approximation to use Δt appropriate to the principal tidal frequency χ_{2200} or to the double instantaneous synodic frequency (18). However inside the 1:1 resonance, Δt would correspond to the libration frequency χ_{lib} which may be very different from the usual tidal frequencies for nonsynchronous rotation. This circumstance may change the value of Δt and therefore of \mathcal{Z} noticeably.

6.4 Tidal dissipation and the point of stall

Be mindful that on the right-hand side of Eq. (61) we have the tidal torque *averaged over the orbit* (see Appendix B for details). Similarly, the second term on the left-hand side of (61) is the permanent-figure torque *averaged over the orbit*—see expression (52).¹⁷ Therefore Eq. (61) renders us the behaviour of η over times longer than the orbital period. This is acceptable, because the orbital period is much shorter than the timescales of our interest. Specifically, it is much shorter than P , see Eq. (58).

Without dissipation, E is conserved during oscillations of η , while in the presence of weak dissipation E changes slowly. Indeed, multiplying both sides of (61) by $\dot{\eta}$ and making use of (54), we arrive at

¹⁷ For the first term, $C\dot{\eta}$, the caveat about orbital averaging is unimportant, because $\dot{\eta}$ bears no dependence upon the mean anomaly.

$$\begin{aligned} \frac{dE}{dt} = & -\mathcal{Z} \left[\dot{\eta}^2 \left(1 + \frac{15}{2} e^2 \right) - 6n \dot{\eta} e^2 \left(1 + \frac{121}{16} e^2 \right) \right] + O(e^6) \\ & + O(i^2/Q) + O(en\Delta t/Q). \end{aligned} \quad (63)$$

Thus we see that a dissipative tidal torque influences the spin, while causing changes also in the first integral E (which is *not* identical to the actual kinetic energy). Considering equation (54) and noting that the cosine term is periodic, we see that weak tidal dissipation will cause a change in $\dot{\eta}^2$ over time scales long compared to P .

For nonsynchronous rotation, weak dissipation causes both a slow secular change and a small oscillation. The latter arises from the oscillating part of $\dot{\eta}$ as η circulates or librates.

Goldreich (1966) also averages over the period P . For libration, the so-averaged rate of change of E is

$$\langle dE/dt \rangle_P = -\frac{\mathcal{Z} W}{P} \left(1 + \frac{15}{2} e^2 \right) + O(e^6) + O(i^2/Q) + O(en\Delta t/Q), \quad (64)$$

where the positive definite quantity W is introduced via (60). The negative $\langle dE/dt \rangle_P$ for libration means that the maximum $\dot{\eta}^2$ at $\cos 2\eta = 1$ in Eq. (54) is decreasing and the free libration damps with time. Libration evolves toward synchronous rotation with $\dot{\eta} = 0$ and η equal to 0 or to π .

The $\dot{\eta}$ term causes the damping of the libration amplitude. When the amplitude is sufficiently small, its decrease obeys the exponential law $\exp(-D_L t)$ with

$$D_L = \frac{\mathcal{Z}}{2C} \left(1 + \frac{15}{2} e^2 \right). \quad (65)$$

As we emphasised in the paragraph after equation (61), inside the 1:1 resonance the time delay Δt showing up in the expression for \mathcal{Z} should be appropriate to the libration frequency χ_{lib} which may differ greatly from the usual tidal frequencies for nonsynchronous spin. This choice of Δt will influence the value of \mathcal{Z} .

For circulation, averaging of Eq. (63) over one orbital period leads to

$$\begin{aligned} \langle dE/dt \rangle_P = & -\frac{\mathcal{Z}}{P} \left[W \left(1 + \frac{15}{2} e^2 \right) - 12\pi n e^2 \left(1 + \frac{121}{16} e^2 \right) \operatorname{sgn} \langle \dot{\eta} \rangle_P \right], \\ & + O(e^6) + O(i^2/Q) + O(en\Delta t/Q) \end{aligned} \quad (66)$$

where we have used definition (59a) for P and definition (60) for W .

While $W = P \langle \dot{\eta}^2 \rangle_P$ is positive definite, the second term on the right-hand side of Eq. (66) can, for circulation, have either sign. In the case when its sign is positive, the expression (66) for $\langle dE/dt \rangle_P$ will vanish for W equal to

$$W_{stall} = 12\pi n e^2 \left(1 + \frac{e^2}{16} \right) + O(e^6) + O(i^2/Q) + O(en\Delta t/Q). \quad (67)$$

No matter whether W begins its evolution with an initial value larger or smaller than W_{stall} , it will never cross W_{stall} .

Another important value of W is the one corresponding to the boundary between libration and circulation, W_b . Below we shall obtain its value and shall explain that for $W_b < W_{stall}$ there exists a positive value of $\langle \dot{\eta} \rangle_P$ at which the evolution of a circulating $\langle \dot{\eta} \rangle_P$ should stall.

6.5 The free-libration frequency

Exploring librations, we start out with the small-amplitude case. Replacing $\sin 2\eta$ with 2η in Eq. (53), we write down the frequency for *small* librations of η :

$$\chi_{lib-max} = \frac{2\pi}{P_{min}} = \left[3 \frac{B-A}{C} \frac{M_{sec}}{M_{sec} + M_{prim}} G_{200}(e) \right]^{1/2} n. \quad (68)$$

In many realistic situations, the condition $\frac{B-A}{C} \frac{M_{sec}}{M_{sec} + M_{prim}} \ll 1$ is fulfilled,¹⁸ which ensures that $\chi_{lib-max} \ll n$. Larger amplitudes increase the period P and render a smaller frequency χ_{lib} , so expression (68) gives the smallest librating period and the *largest* frequency

$$\chi_{lib-max} = \max \chi_{lib}. \quad (69)$$

The linear $\dot{\eta}$ term in (61) will alter the frequency, but for slow tide-caused evolution the correction will be small, so (68) still will serve well as an approximation for the maximal frequency of libration. On all these grounds, we now accept that for both small-amplitude and large-amplitude librations the inequality

$$\chi_{lib} \ll n \quad (70)$$

holds. This justifies, *a posteriori*, the assertion made after (59b).

In the literature on the physical libration of the Moon, the expression for the free-libration frequency is ubiquitous, though often without the mass factor. Versions of the expressions for η_{bias} and the free-libration damping rate appeared in Williams et al. (2001). That paper, though, did not rely on the corrected MacDonald model (constant time delay), but simply used separate Q s for libration and orbital frequencies.

6.6 The boundary between circulation and libration

The boundary between circulation and the resonance zone corresponds to a value $W = W_b$. To find it, we combine (54) with (56) and arrive at

$$\frac{1}{2} C \dot{\eta}^2 = E_b (1 + \cos 2\eta), \quad (71a)$$

which is the same as

$$\dot{\eta}^2 = 4 \frac{E_b}{C} \cos^2 \eta. \quad (71b)$$

Insertion of the resulting expression for $\dot{\eta}$ into (60) entails:

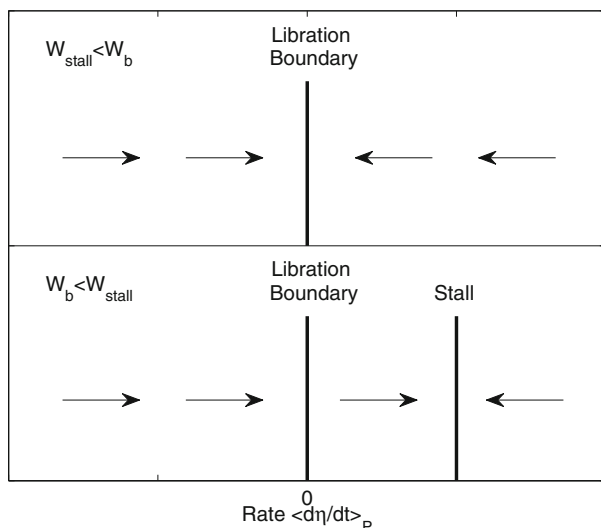
$$W_b = 8 \sqrt{\frac{E_b}{C}} \int_0^{\pi/2} \cos \eta \, d\eta = 8 \sqrt{\frac{E_b}{C}} = 4n \left[3 \frac{B-A}{C} \frac{M_{sec}}{M_{sec} + M_{prim}} G_{200}(e) \right]^{1/2}, \quad (72)$$

comparison whereof to (68) yields another expression for the boundary value:

$$W_b = 4 \chi_{lib-max}.$$

¹⁸ This condition is satisfied safely if the primary is a planet (like Mercury) or a large satellite (like our Moon). Its fulfilment is not guaranteed, though, for small satellites (like Phobos or Hyperion).

Fig. 1 Three possible scenarios of evolution of $\langle \dot{\eta} \rangle_p$. Arrows on the top diagram illustrate the evolution of circulating $\langle \dot{\eta} \rangle_p$ toward the libration boundary for $W_{stall} < W_b$. Arrows on the left of the bottom diagram depict the evolution of negative $\langle \dot{\eta} \rangle_p$ toward the libration boundary. Arrows in the midst and on the right of the bottom diagram show the evolution of positive $\langle \dot{\eta} \rangle_p$ toward the stall point for $W_b < W_{stall}$



While W is continuous across the boundary, P has a logarithmic singularity, as was mentioned in the paragraph after equation (56). As P diverges, the evolution rate of $\langle dE/dt \rangle_p$, given by (66), vanishes at the boundary. Nonetheless a small perturbation allows the boundary to be crossed—for more on this, see the two paragraphs after formula (22) in Goldreich (1966).

6.7 Three regimes

If $W_{stall} < W_b$, then there is no stall point in the region of circulation ($W_b < W$). So W evolves towards W_b , while $\langle \dot{\eta} \rangle_p$ of either sign evolves towards zero at the libration/circulation boundary. Figure 1a illustrates the evolution of $\langle \dot{\eta} \rangle_p$. Goldreich (1966) comments that the boundary will be crossed, the free libration will damp, and the rotation will evolve toward the synchronous state. The synchronous state has a zero $\dot{\eta}$, with η biased slightly off¹⁹ of either 0 or π . The figure does not show the libration region.²⁰

If $W_b < W_{stall}$, then a stall point exists in the circulating region ($W_b < W$). All in all, Fig. 1b summarises the following three cases:

- A. For an initially negative $\langle \dot{\eta} \rangle_p$, circulation is taking place. We have $W_b < W$, but the stall point is never located on the negative $\langle \dot{\eta} \rangle_p$ side of zero. The rate $\langle dE/dt \rangle_p$ is negative, as can be seen from (63). We then have a slow decrease of the three quantities:²¹ $E \rightarrow E_b$, $W \rightarrow W_b$, and $\langle \dot{\eta} \rangle_p \rightarrow 0$. This decrease takes the system to the circulation/libration boundary. Goldreich (1966) commented that it is ambiguous whether $\langle \dot{\eta} \rangle_p$ would cross zero and proceed to increase or whether the boundary would be crossed

¹⁹ The value π shows up because of the factor 2 accompanying η when this variable enters $\sin 2\eta$ in the equation for the torque and $\cos 2\eta$ in the expression E . The tiny bias off $\eta = 0$ or off $\eta = \pi$, given by (62), will emerge due to dissipation.

²⁰ A figure with W rather than $\langle \dot{\eta} \rangle_p$ would show the libration region, along with the circulating region. On such a figure, though, positive and negative values of $\langle \dot{\eta} \rangle_p$ would overlap, so that the one-sided nature of the stall would not be evident.

²¹ Be mindful that E remains a constant over a cycle of P , except for a tiny amount of dissipation during a cycle.

passing into the libration region followed by damping of the libration and by evolution toward the synchronous state.

Be mindful that, while the first integral E is decreasing in the circulation case, the actual kinetic energy of rotation is increasing. Indeed, when the negative $\langle \dot{\eta} \rangle_P$ is evolving towards zero, the spin rate $\dot{\theta}$ is growing, and so is the rotational energy $C\dot{\theta}^2/2$.

- B.** For an initially positive $\langle \dot{\eta} \rangle_P$ lying between the circulation/libration boundary and the stall point, i.e., for $W_b < W < W_{stall}$, the rate $\langle dE/dt \rangle_P$ is positive, and E increases. The quantities W , $\langle \dot{\eta}^2 \rangle_P$ and $\langle \dot{\eta} \rangle_P$ will evolve to larger values until the evolution stalls as W approaches W_{stall} from below.

As the positive $\dot{\eta}$ is increasing, so are the spin rate $\dot{\theta}$ and the rotational energy.²²

- C.** For an initially positive $\langle \dot{\eta} \rangle_P$ beyond the stall point we have: $W_b < W_{stall} < W$. Then $\langle dE/dt \rangle_P$ is negative and E is slowly decreasing.²³ Likewise, W , $\langle \dot{\eta}^2 \rangle_P$ and $\langle \dot{\eta} \rangle_P$ evolve to lower values until the evolution stalls as W approaches W_{stall} from above. The decrease of the positive $\dot{\eta}$ renders a decrease in the rotation rate $\dot{\theta}$ and thus leads to damping of the kinetic energy of rotation.

Calculation of the frequency at the stall point $\langle \dot{\eta}_{stall} \rangle_P$ would require two steps: first, reversing the elliptic integral computation, starting with W_{stall} ; and second, deriving E/C , P_{stall} and $\langle \dot{\eta}_{stall} \rangle_P$.

At the stall point, formulae (59a) and (60) acquire the form of

$$P_{stall} = \frac{2\pi}{\langle \dot{\eta}_{stall} \rangle_P} \quad \text{and} \quad W_{stall} = P_{stall} \langle \dot{\eta}_{stall}^2 \rangle_P, \quad (73)$$

correspondingly. In combination with (67), this entails:

$$\frac{\langle \dot{\eta}_{stall}^2 \rangle_P}{\langle \dot{\eta}_{stall} \rangle_P} = 6n e^2 \left(1 + \frac{e^2}{16} \right), \quad (74)$$

whence we once again note that the frequency $\langle \dot{\eta}_{stall} \rangle_P$ must be positive. The comparison of W_{stall} with W_b in the inequalities mentioned in the above items **A**, **B**, **C** is then equivalent to comparing $\langle \dot{\eta}_{stall}^2 \rangle_P / \langle \dot{\eta}_{stall} \rangle_P$ with $(2/\pi) \chi_{lib-max}$ or to comparing²⁴ $3\pi e^2 (1 + e^2/16)$ with $\left[3 \frac{B-A}{C} \frac{M_{Earth}}{M_{Earth} + M_{Moon}} G_{200}(e) \right]^{1/2}$.

A comment on Eq. (74) would be in order. Since oscillations of η result from the existence of the permanent triaxiality, evolution of η becomes smooth in the $A = B$ limit. Averaging becomes unnecessary, so $\langle \dot{\eta}_{stall}^2 \rangle_P$ becomes simply $\dot{\eta}_{stall}^2$, while $\langle \dot{\eta}_{stall} \rangle_P$ becomes $\dot{\eta}_{stall}$. This way, in the oblate-body case considered back in Sect. 5, Eq. (74) acquires the form of $\dot{\eta}_{stall} = 6n e^2 (1 + e^2/16)$, which agrees with (44).

6.8 Application to the Moon

When Goldreich (1966) applied his inequality expressions to the Moon, he found that the stall point would have interrupted evolution of rotation from faster spin to synchronous

²² In the first integral (54), the kinetic-energy-like part is given by $C\dot{\eta}^2/2 = (C/2)(\dot{\theta} - n)^2 = (C/2)(\dot{\theta}^2 - 2n\dot{\theta} + n^2)$. When the spin accelerates, the additional kinetic energy is borrowed from the orbital motion. Because of the dissipation, the overall spin + orbit energy must nevertheless decrease. (The orbital energy includes a kinetic part and the $-GM/r$ potential term).

²³ Just as in item **A**, here E stays virtually unchanged over P .

²⁴ The latter may also be expressed as comparison of $3\pi e^2 \left(1 + \frac{21}{16} e^2 \right)$ with $\left[3 \frac{B-A}{C} \frac{M_{Earth}}{M_{Earth} + M_{Moon}} \right]^{1/2}$.

rotation. Here we have repeated his study, though with the corrected average torque (43) instead of (34). For $\frac{B-A}{C} = 2.278 \times 10^{-4}$ (Williams and Boggs 2012), $e = 0.0549$, and $\frac{M_{Earth}}{M_{Earth} + M_{Moon}} = 0.98785$, the libration period turns out to be 38 times the orbital period, while W_{stall} is 10% larger than W_b . This renders a value of W_{stall}/W_b much smaller than the one found by Goldreich (1966), and the difference is mainly due to our use of the corrected average torque (43). Despite the so-different value of W_{stall}/W_b , despinning of the Moon would still be interrupted by a stall. In principle, a tidal spin-up scenario remains an option too, though this option does not look probable.

Goldreich (1966) noted that the lunar orbital eccentricity is changing. To make the stall point disappear, i.e., to ensure that $W_{stall} < W_b$, the eccentricity of the Moon would need to be less than 95% of its present value. While we lack data on the ancient evolution of the lunar eccentricity, the modern eccentricity rate of about $2 \times 10^{-11} \text{ year}^{-1}$ has been reliably determined through analysis of the Lunar Laser Ranging data (Williams et al. 2001; Williams and Boggs 2009). For the measured eccentricity rate, the eccentricity would be small enough to prevent a stall prior to 1.4×10^8 year ago. The Moon has clearly been a satellite of the Earth for billions of years, and the capture into the synchronous spin state should have occurred very early in its history.

For the Moon as it exists today, the free libration in longitude has a 2.9 year period. The damping time is four orders of magnitude longer (Williams et al. 2001), so evolution of this libration is slow. Despite the damping time being short compared to the lunar age, the Moon has a small free libration amplitude of $1.3''$ (Rambaux and Williams 2011). There has been geologically recent stimulation, probably due to resonance crossing (Eckhardt 1993).

The ambiguity of evolution of the negative $\langle \dot{\eta} \rangle_p$ past the circulation/libration boundary deserves a comment. For Mercury, Makarov (2012) finds that a more realistic tidal dissipation model than the corrected MacDonald torque strongly changes computations of the evolution of planetary spin rate near the 3 : 2 and higher spin-orbit resonances.

7 Conclusions

In the article thus far, we have provided a detailed explanation of how the empirical MacDonald model can be derived from a more accurate and comprehensive Darwin–Kaula theory of bodily tides. We have demonstrated that the derivation hinges on a key assertion that the quality factor Q of the primary should be inversely proportional to the tidal frequency. This crucial circumstance was missed by MacDonald (1964), who made his theory inherently self-contradictory by setting the quality factor to be a frequency-independent constant.

We have corrected this oversight in the MacDonald approach, and have developed an appropriate correction to Goldreich's model of spin dynamics and evolution near the 1:1 spin-orbit resonance. Although we got different numbers, qualitatively the main conclusion by Goldreich (1966) stays unaltered: when an oblate body's spin is evolving toward the resonance, vanishing of the average tidal torque still implies rotation slightly faster than resonant, while synchronicity requires a small compensating torque. For a triaxial body, the picture gets more complex due to the emergence of a triaxiality-caused torque. (While the oblate case is appropriate for spinning gaseous or liquid planets and moons, the triaxial case applies to rocky objects).

Goldreich (1966) linked the possible trapping of a body in the synchronous state during its tidal evolution of rotation to its triaxiality. In light of the correction of expression (36) to (44), that limiting condition between the triaxiality and e^2 must change. After capture into

the synchronous state, the tidal torque is compensated by a triaxiality torque by aligning the principal axis associated with the smallest moment of inertia slightly off of the mean direction to the external body. A constant is thereby introduced into the physical libration in longitude.

Setting the tidal quality factor to scale as inverse frequency is incompatible with the actual dissipative properties of realistic mantles and crusts. Nevertheless, after the aforementioned correction is implemented, both the corrected MacDonald description of tides and the dynamical theory based thereon remain valuable toy models capable of providing a good qualitative handle on tidal dynamics over not too long timescales. Specifically, this approach renders a simple qualitative description of the interplay between the tidal torque and the triaxiality-caused torque exerted on a body near the 1:1 spin-orbit resonance.

It should also be remembered that the stability of pseudosynchronous rotation hinges upon the dissipation model employed. [Makarov and Efroimsky \(2012\)](#) have found that a more realistic tidal dissipation model than the corrected MacDonald torque makes pseudosynchronous rotation unstable. Finally, pseudosynchronism becomes impossible when the triaxiality is too large (see subsection 6.7, specifically footnote 24).

Acknowledgments We are indebted to Sylvio Ferraz-Mello for numerous fruitful discussions on the theory of bodily tides and for referring us to the paper by [Rodríguez et al. \(2008\)](#). We also express our deep thanks to Anthony Dobrovolskis whose review of our manuscript was very comprehensive and extremely helpful. A portion of the research described in this paper was carried out at the Jet Propulsion Laboratory of the California Institute of Technology, under a contract with the National Aeronautics and Space Administration. Government sponsorship acknowledged.

Appendices

Appendix A: The tidal-torque vector and its components in spherical coordinates

A secondary with spherical coordinates $\vec{r}^* = (r^*, \lambda^*, \phi^*)$ and mass M_{sec}^* raises a tidal bulge on the primary. The gravitational attraction between the tidal bulge and a secondary at $\vec{r} = (r, \lambda, \phi)$ with mass M_{sec} causes equal but opposite torques on the primary and the secondary. For the external tidal potential $U(\vec{r})$, the torque components depend on the partial derivatives of the potential U along great circle arcs.

To calculate these expressions, let us recall some basics. The torque \vec{T} wherewith the primary is acting on the secondary is given by the cross-product

$$\vec{T} = \vec{r} \times \vec{\mathcal{F}}, \quad (75)$$

$\vec{\mathcal{F}}$ being the tidal force exerted by the primary on the secondary. This force is given by

$$\vec{\mathcal{F}} = -M_{sec} \frac{\partial U(\vec{r})}{\partial \vec{r}} = -M_{sec} \left(\frac{\partial U}{\partial r} \hat{e}_r + \frac{1}{r} \frac{1}{\cos \phi} \frac{\partial U}{\partial \lambda} \hat{e}_\lambda - \frac{1}{r} \frac{\partial U}{\partial \phi} \hat{e}_\phi \right), \quad (76)$$

\hat{e}_r , \hat{e}_λ , \hat{e}_ϕ being the unit vectors of a spherical coordinate system associated with the primary's equator and corotating with it.²⁵ Insertion of (76) into (75) results in

²⁵ Were we using the polar angle $\varphi = \pi/2 - \phi$ instead of the latitude ϕ , the right-handed triple of unit vectors would be: \hat{e}_r , \hat{e}_φ , \hat{e}_λ ("radial – south – east"), while the gradient would read as $\frac{\partial U(\vec{r})}{\partial \vec{r}} = \frac{\partial U}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial U}{\partial \varphi} \hat{e}_\varphi + \frac{1}{r} \frac{1}{\sin \varphi} \frac{\partial U}{\partial \lambda} \hat{e}_\lambda$.

As we are employing the latitude, the right-handed triple changes to \hat{e}_r , \hat{e}_λ , \hat{e}_ϕ ("radial – east – north"), and the gradient becomes $\frac{\partial U(\vec{r})}{\partial \vec{r}} = \frac{\partial U}{\partial r} \hat{e}_r + \frac{1}{r} \frac{1}{\cos \phi} \frac{\partial U}{\partial \lambda} \hat{e}_\lambda - \frac{1}{r} \frac{\partial U}{\partial \phi} \hat{e}_\phi$.

$$\vec{\mathbf{T}} = -M_{sec} \left(0 \hat{\mathbf{e}}_r + \frac{\partial U}{\partial \phi} \hat{\mathbf{e}}_\lambda + \frac{1}{\cos \phi} \frac{\partial U}{\partial \lambda} \hat{\mathbf{e}}_\phi \right), \quad (77)$$

The torque $\vec{\mathbf{\tau}}$ wherewith the secondary is acting on the primary will be the negative of (77):

$$\vec{\mathbf{\tau}} = M_{sec} \left(0 \hat{\mathbf{e}}_r + \frac{\partial U}{\partial \phi} \hat{\mathbf{e}}_\lambda + \frac{1}{\cos \phi} \frac{\partial U}{\partial \lambda} \hat{\mathbf{e}}_\phi \right) \quad (78)$$

Its east component, the one aimed along $\hat{\mathbf{e}}_\lambda$, is parallel to the primary's equatorial plane. The north component, aimed along $\hat{\mathbf{e}}_\phi$, is tangent to the meridian. In our paper however we employ the projection of the torque vector onto the primary's spin axis, i.e., the component τ_z orthogonal to the equator plane. This component is $\cos \phi$ times the north component:

$$\mathcal{T}_z = M_{sec} \frac{\partial U}{\partial \lambda}. \quad (79)$$

This torque component, Eq. (6), slows down the rotation rate of the primary. The decelerating torque acting on the secondary has an opposite sign and is rendered by (5).

The east component and the projection of the north component onto the equator plane act to change the orientation of the primary's spin axis, a subject we shall not pursue in this paper.

Appendix B: Calculation of the average torque within the corrected MacDonald model

To calculate the orbital average of the tidal torque acting on a librating secondary obeying the corrected MacDonald tidal model, substitute (38) and (18) into (22), and then choose $\alpha = -1$ (and recall that, for $\alpha = -1$, the integral rheological parameter \mathcal{E} is simply the time lag: $\mathcal{E} = \Delta t$). An equivalent option would be to employ (23) directly. This will lead us to the following expression for the torque:

$$\begin{aligned} \mathcal{T}_z &= -\frac{3}{2} G M_{sec}^2 k_2 \frac{R^5}{r} \mathcal{E} \chi \operatorname{sgn}(\dot{\theta} - \dot{\nu}) + O(i^2/Q) + O(en\Delta t/Q) + O(Q^{-3}) \\ &= -\frac{3}{2} G M_{sec}^2 k_2 \frac{R^5}{r} \frac{2}{6} \mathcal{E} |\dot{\theta} - \dot{\nu}| \operatorname{sgn}(\dot{\theta} - \dot{\nu}) + O(i^2/Q) + O(en\Delta t/Q) + O(Q^{-3}) \\ &= -3 G M_{sec}^2 k_2 \frac{R^5}{r} \mathcal{E} (\dot{\theta} - \dot{\nu}) + O(i^2/Q) + O(en\Delta t/Q) + O(Q^{-3}), \end{aligned} \quad (80)$$

and for its average over one orbiting cycle:

$$\begin{aligned} \langle \mathcal{T}_z \rangle &= -\frac{3 G M_{sec}^2 k_2 \mathcal{E}}{R} \left\langle (\dot{\theta} - \dot{\nu}) \frac{R^6}{r^6} \right\rangle + O(i^2/Q) + O(Q^{-3}) + O(en\Delta t/Q) = \\ &= -\frac{3 G M_{sec}^2 k_2 \mathcal{E}}{R} \dot{\theta} \left\langle \frac{R^6}{r^6} \right\rangle + \frac{3 G M_{sec}^2 k_2 \mathcal{E}}{R} \left\langle \dot{\nu} \frac{R^6}{r^6} \right\rangle + O(i^2/Q) + O(Q^{-3}) \\ &\quad + O(en\Delta t/Q) \\ &= -\frac{3 G M_{sec}^2 k_2 \mathcal{E}}{R} \dot{\theta} \frac{R^6}{a^6} (1 - e^2)^{-9/2} \frac{1}{2\pi} \int_0^{2\pi} (1 + e \cos \nu)^4 d\nu \end{aligned} \quad (81a)$$

$$\begin{aligned}
& + \frac{3GM_{\text{sec}}^2 k_2 \varepsilon}{R} n \frac{R^6}{a^6} (1-e^2)^{-6} \frac{1}{2\pi} \int_0^{2\pi} (1+e \cos v)^6 dv + O(i^2/Q) + O(Q^{-3}) \\
& + O(en\Delta t/Q).
\end{aligned} \tag{81b}$$

Evaluation of the integrals is straightforward and entails (39–42).

References

- Abramowitz, M., Stegun, I.A. (ed.): Handbook of mathematical functions. The National Bureau of Standards, Applied Mathematics Series, Vol. 55, pp. 332. US Government Printing Office, Washington DC (1972)
- Arfken, G.B., Weber, H.J.: Mathematical Methods for Physicists. pp. 723 Academic Press, Boston (1995)
- Castillo-Rogez, J.C., Efroimsky, M., Lainey, V.: The tidal history of Iapetus. Dissipative spin dynamics in the light of a refined geophysical model. *J. Geophys. Res. Planets* **116**(E9), id. E09008 (2011). doi:[10.1029/2010JE003664](https://doi.org/10.1029/2010JE003664)
- Correia, A.C.M., Laskar, J.: Mercury's capture into the 3/2 spin-orbit resonance as a result of its chaotic dynamics. *Nature* **429**, 848–850 (2004)
- Correia, A.C.M., Laskar, J.: Mercury's capture into the 3/2 spin-orbit resonance, including the effect of core-mantle friction. *Icarus* **201**, 1–11 (2009)
- Correia, A.C.M., Boué, G., Laskar, J.: Pumping the eccentricity of exoplanets by tidal effect. Submitted to the *Astrophys. J. Lett.* arXiv:1111.5486 (2011)
- Darwin, G.H.: On the precession of a viscous spheroid and on the remote history of the Earth. *Philos. Trans. R. Soc. Lond.* **170**, 447–530 (1879). <http://www.jstor.org/view/02610523/ap000081/00a00010/>
- Darwin, G.H.: On the secular change in the elements of the orbit of a satellite revolving about a tidally distorted planet. *Philos. Trans. R. Soc. Lond.* **171**, 713–891. <http://www.jstor.org/view/02610523/ap000082/00a00200>
- Dobrovolskis, A.: Spin states and climates of eccentric exoplanets. *Icarus* **192**, 1–23 (2007)
- Eckhardt, D.H.: Passing through resonance: the excitation and dissipation of the lunar free libration in longitude. *Celest. Mech. Dyn. Astron.* **57**, 307–324 (1993)
- Efroimsky, M.: Bodily tides near spin-orbit resonances. *Celest. Mech. Dyn. Astron.* **112**, 283–330 (2012a). doi:[10.1007/s10569-011-9397-4](https://doi.org/10.1007/s10569-011-9397-4). Extended version available at: arXiv:1105.6086
- Efroimsky, M.: Tidal dissipation compared to seismic dissipation: in small bodies, earths, and superearths. *Astrophys. J.* **746**(2), article id 150 (2012b). doi:[10.1088/0004-637X/746/2/150](https://doi.org/10.1088/0004-637X/746/2/150). arXiv:1105.3936
- Efroimsky, M., Lainey, V.: The physics of bodily tides in terrestrial planets, and the appropriate scales of dynamical evolution. *J. Geophys. Res. Planets* **112**, article id. E12003 (2007). doi:[10.1029/2007JE002908](https://doi.org/10.1029/2007JE002908)
- Efroimsky, M., Makarov, V.V.: Applicability limitations of a popular formula for the tidal torque. Submitted to the *Astrophys. J.* (2012). arXiv:1209.1615
- Efroimsky, M., Williams, J.G.: Tidal torques. A critical review of some techniques. *Celest. Mech. Dyn. Astron.* **104**, 257–289 (2009). doi:[10.1007/s10569-009-9204-7](https://doi.org/10.1007/s10569-009-9204-7). arXiv:0803.3299
- Ferraz-Mello, S., Rodríguez, A., Hussmann, H.: Tidal friction in close-in satellites and exoplanets: the Darwin theory re-visited. *Celest. Mech. Dyn. Astron.* **101**, 171–201 (2008)
- Gerstenkorn, H.: Über Gezeitenreibung beim Zweikörperproblem. *Zeitschrift für Astrophysik* **36**, 245–274 (1955)
- Goldreich, P.: Final spin states of planets and satellites. *Astron. J.* **71**, 1–7 (1966)
- Goldreich, P., Peale, S.: Spin-orbit coupling in the Solar system. *Astron. J.* **71**, 425–438 (1966)
- Gooding, R.H., Wagner, C.A.: On the inclination functions and a rapid stable procedure for their evaluation together with derivatives. *Celest. Mech. Dyn. Astron.* **101**, 247–272 (2008)
- Heller, R., Leconte, J., Barnes, R.: Tidal obliquity evolution of potentially habitable planets. *Astron. Astrophys.* **528**, (2011) article id.A27. doi:[10.1051/0004-6361/201015809](https://doi.org/10.1051/0004-6361/201015809)
- Hut, P.: Tidal evolution in close binary systems. *Astron. Astrophys.* **99**, 126–140 (1981)
- Kaula, W.M.: Analysis of gravitational and geometric aspects of geodetic utilisation of satellites. *Geophys. J.* **5**, 104–133 (1961)
- Kaula, W.M.: Tidal dissipation by solid friction and the resulting orbital evolution. *Rev. Geophys.* **2**, 661–684 (1964)
- Kaula, W.M.: Theory of Satellite Geodesy: Applications of Satellites to Geodesy. Blaisdell Publishing Co, Waltham MA. (Re-published in 2006 by Dover. ISBN: 0486414655) (1966)
- Kaula, W.M.: An Introduction to Planetary Physics. Wiley, NY (1968)

- Laskar, J., Correia, A.: The rotation of extra-solar planets. *Extrasolar Planets: Today and Tomorrow*. ASP Conference Series **321**, 401–409 (2004)
- MacDonald, G.J.F.: Tidal friction. *Rev. Geophys.* **2**, 467–541 (1964)
- Makarov, V.V.: Conditions of passage and entrapment of terrestrial planets in spin-orbit resonances. *Astrophys. J.* **752** article id. 73 (2012). arXiv:1110.2658
- Makarov, V.V., Efroimsky, M.: No pseudosynchronous rotation for terrestrial planets and moons. Submitted to the *Astrophys. J.* (2012). arXiv:1209.1616
- Mignard, F.: The evolution of the lunar orbit revisited. I. *Moon Planets* **20**, 301–315 (1979)
- Mignard, F.: The evolution of the lunar orbit revisited. II. *Moon Planets* **23**, 185–201 (1980)
- Mignard, F.: Evolution of the Martian satellites. *Mon. Notices R. Astron. Soc.* **194**, 365–379 (1981)
- Murray, C.D., Dermott, S.F.: *Solar System Dynamics*. Cambridge University Press, Cambridge (1999)
- Petit, G., Luzum, B. (eds.): *IERS Conventions* (2010). IERS Technical Note 36, Verlag des Bundesamts für Kartographie und Geodäsie, Frankfurt am Main 2010. <http://www.iers.org/IERS/EN/Publications/TechnicalNotes/tn36.html>
- Rambaux, N., Williams, J.G.: The Moon's physical librations and determination of their free modes. *Celest. Mech. Dyn. Astron.* **109**, 85–100 (2011). doi:[10.1007/s10569-010-9314-2](https://doi.org/10.1007/s10569-010-9314-2)
- Rodríguez, A., Ferraz-Mello, S., Hussmann, H.: Tidal friction in close-in planets. In: Sun, Y.S., Ferraz-Mello, S., Zhou, J.L. (eds.) *Exoplanets: Detection, Formation and Dynamics*. Proceedings of the IAU Symposium No 249, pp. 179–186 (2008). doi:[10.1017/S174392130801658X](https://doi.org/10.1017/S174392130801658X)
- Singer, S.F.: The origin of the Moon and geophysical consequences. *Geophys. J. R. Astron. Soc.* **15**, 205–226 (1968)
- Standish, E.M., Williams, J.G.: Orbital ephemerides of the Sun, Moon, and planets, Chapter 8 of the Explanatory Supplement to the Astronomical Almanac, Ed. by P. K. Seidelmann (2012) (in press)
- Williams, J.G., Boggs, D.H.: Lunar core and mantle. What Does LLR See? In: Schilliak, S. (ed.) *Proceedings of the 16th International Workshop on Laser Ranging*, held on 12–17 October 2008 in Poznan, Poland, pp. 101–120 (2009)
- Williams, J.G., Boggs, D.H.: In preparation (2012)
- Williams, J.G., Boggs, D.H., Yoder, C.F., Ratcliff, J.T., Dickey, J.O.: Lunar rotational dissipation in solid-body and molten core. *J. Geophys. Res. Planets*, **106**(E11), 27933–27968 (2001). doi:[10.1029/2000JE001396](https://doi.org/10.1029/2000JE001396)